Math 347: Homework 5
Due on: Oct. 17, 2018

1. Let S be a bound subset of $\mathbb{R}$. Suppose that x_n and y_n are two sequences, such that $x_n \to \sup(S)$ and $y_n \to \inf(S)$. Prove that $\lim x_n + y_n \in S$.

2. For each set S below, find a sequence in S converging to $\sup(S)$ and a sequence converging to $\inf(S)$.

   (i) $S = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$;

   (ii) $S = \{\frac{2+(-1)^n}{n} \mid n \in \mathbb{N}\}$.

3. Prove that the least upper bound property for $\mathbb{R}$ holds if and only if the greaster lower bound property for $\mathbb{R}$ holds.

4. Let $X_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$. Prove that the sequence x_n converges.

5. *The nested interval property.* Let $\{I_n\}$ be a sequence of closed intervals, with I_n of length d_n, such that $I_{n+1} \subset I_n$ and $d_n \to 0$.

   The *nested interval property*: for such a sequence, there is exactly one point that belongs to all I_n.

   Prove the following:

   (i) the completeness axiom implies the nested interval property;

   (ii) the nested interval property implies the completeness axiom.